Assignment 1

This homework is due Friday Jan 30.

There are total 40 points in this assignment. 35 points is considered 100%. If you go over 35 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

(1) [10pt] Perform the required calculations and express your answer as $a + bi$ (or just as a if the answer is a real number).

(a) i^{371}.
(b) $\frac{1}{i^5}$.
(c) $\mathrm{Re}(-i)$.
(d) $\mathrm{Im}(2015)$.
(e) $(1 - i)^3$.
(f) $(i + 1)^8$.
(g) $(5 - 2i)(7i + 4)$.
(h) $\mathrm{Im}\left(\frac{1-2i}{4+3i}\right)$.
(i) $\frac{(4-i)(1-3i)}{1+2i}$.
(j) $\overline{(1 + i\sqrt{3})(i + \sqrt{3})}$.

[*Hint* for item 1f: $A^8 = ((A^2)^2)^2$.]

(2) [10pt] Evaluate the following:

(a) $\overline{(1 + i)(2 + i)}(3 + i)$.
(b) $(3 + i)/\overline{(2 + i)}$.
(c) $\mathrm{Im}((1 - i)^2)$.
(d) $\frac{1+2i}{3+4i} - \frac{4-3i}{2-i}$.
(e) $\mathrm{Re}((x - yi)^2)$.
(f) $\mathrm{Im}\left(\frac{1}{x-iy}\right)$.
(g) $\mathrm{Im}((x + iy)(-x + iy))$.
(h) $\mathrm{Im}((x + iy)^3)$.

(3) [5pt]
(a) Show that $z\bar{z}$ is always a real number.
(b) Verify that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

(4) [10pt] Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree n.
(a) Suppose that $a_n, \ldots, a_1, a_0$ are all real. Show that if z_0 is a complex root of P then $\overline{z_0}$ is also a root. (In other words, non-real complex roots split into pairs of conjugates.)
(b) Suppose polynomial $P = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ with complex coefficients has the above property, i.e. if z_0 is its root then $\overline{z_0}$ also is. Prove that all coefficients of P are, in fact, real. You can take for granted that nth degree polynomial has n complex roots[1] [*Hint:* if $z_1, \ldots, z_n$ are roots of P, then P can be expressed as $P(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$.]
(c) (*Optional, +3pt*) Prove that every polynomial with real coefficients can be factored into a product of linear and quadratic polynomials with real coefficients.

(5) [5pt] Consider matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with a, b real.

For $Z_1 = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}$ and $Z_2 = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$ find $Z_1 Z_2$. Compare to multiplication of complex numbers. Make conclusions (I am not asking for an essay, one sentence will be enough).

[1] We will prove that later in the course.